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FINITE DIMENSIONAL ULTRAMETRIC CONVEXITY

by

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ABSTRACT

Convex sets in a finite dimensional vector space over a nonarchimedean valued field are studied. Ultrametric versions of the classical theorems of Helly and Carathéodory are obtained. Necessary and sufficient conditions in order that a point in a convex set can be separated by a hyperplane are derived.

INTRODUCTION

Throughout, K is a nonarchimedean nontrivially valued complete field, $n \in \mathbb{N}$. A nonempty subset A of K^n is absolutely convex if $x, y \in A$, $\alpha, \beta \in B(0, 1) := \{ \lambda \in K : |\lambda| \leq 1 \}$ implies $\alpha x + \beta y \in A$. The motive for this paper is the following question. Let $A \subset K^n$ be absolutely convex and $x \in K^n \setminus A$. Does there exist an $f \in (K^n)'$ (the dual space of K^n) such that $f(x) \notin f(A)$? (It is easily seen that A is closed with respect to the (unique) norm topology on K^n so that the distance between $\{x\}$ and A is positive.) The answer is yes if K is spherically (= maximally) complete [4]. If, however, K is not spherically complete the answer is, in general, no; Theorem 4.8 offers a characterization. §4 does not depend on §3.

§1 TERMINOLOGY

We shall use the notations and terminology of [2]. For a set $X \subset K^n$ we denote its absolutely convex hull by $\text{co } X$, its K -linear span by $[X]$. Let C, A, B be absolutely convex subsets of K^n . C is the direct sum of A and B (notation $C = A \oplus B$) if $C = A + B (= \{ a+b : a \in A, b \in B \})$ and $A \cap B = (0)$. If $C = A \oplus B$ then $[C] = [A] \oplus [B]$, hence $\dim C = \dim A + \dim B$ (where, for an absolutely convex set S , $\dim S := \dim [S]$). An absolutely convex set $C \subset K^n$ is indecomposable if $C = A \oplus B$, A, B absolutely convex subsets of K^n , implies $A = (0)$ or $B = (0)$.

By dimension arguments one proves easily that each absolutely convex subset of K^n is the direct sum of finitely many indecomposables.

For an absolutely convex set $A \subset K^n$,

$$p_A(x) = \inf \{ |\lambda| : \lambda \in K, x \in \lambda A \}$$

defines its associated seminorm $p_A : [A] \rightarrow [0, \infty)$. Set

$$A^i := \{ x : p_A(x) < 1 \}$$

$$A^e := \{ x : p_A(x) \leq 1 \}$$

Then $A^i \subset A \subset A^e$. A is edged if $A = A^e$. If the valuation of K is discrete each absolutely convex set in K^n is edged. If the valuation of K is dense we have

$$A^i = \bigcup_{|\lambda| < 1} \lambda A, \quad A^e = \bigcap_{|\lambda| > 1} \lambda A$$

So that A is edged if and only if $x \in K^n$, $\lambda x \in A$ for all $\lambda \in K$ with $|\lambda| < 1$ implies $x \in A$.

Normorthogonality [2] can in an obvious way be generalized to orthogonality with respect to a seminorm.

§2 ELEMENTARY SETS

DEFINITION 2.1 Let A be a subset of K^n . A is an elementary set if there exist $x_1, \dots, x_m \in K^n$ and absolutely convex C_1, \dots, C_m in K such that

$A = C_1 x_1 + \dots + C_m x_m$. A is a coelementary set if there exist $f_1, \dots, f_m \in (K^n)'$ and absolutely convex C_1, \dots, C_m in K such that $A = \bigcap_i f_i^{-1}(C_i)$.

We shall prove (Theorem 2.4) that 'elementary' and 'coelementary' are identical notions.

DIMENSION LEMMA 2.2 Let $m \geq 2$ and let

$$A = C_1 x_1 + \dots + C_m x_m$$

$(x_1, \dots, x_m \in K^n, C_1, \dots, C_m \text{ absolutely convex in } K)$ be an elementary set.

If $\dim [x_1, \dots, x_m] < m$ then there is a $j \in \{1, \dots, m\}$ such that

$$A = \sum_{i \neq j} C_i x_i$$

Proof. We may suppose (renumeration) that there exists a $k \in \{2, \dots, m\}$

and $\lambda_1, \dots, \lambda_k \in K \setminus \{0\}$ such that $\lambda_1 x_1 + \dots + \lambda_k x_k = 0$. The sets

$\lambda_1^{-1} C_1, \lambda_2^{-1} C_2, \dots, \lambda_k^{-1} C_k$ are absolutely convex in K hence linearly ordered by inclusion ; say that $\lambda_1^{-1} C_1$ is the smallest among them.

Let $c_1 \in C_1$. From

$$c_1 x_1 = - \sum_{i=2}^k \lambda_i \lambda_1^{-1} c_1 x_i$$

and

$$-\lambda_i \lambda_1^{-1} c_1 \in -\lambda_i \lambda_i^{-1} C_i = C_i \quad (i \in \{2, \dots, k\})$$

we obtain

$$C_1 x_1 \subset \sum_{i=2}^k C_i x_i \subset C_2 x_2 + \dots + C_m x_m$$

It follows that

$$A = C_2 x_2 + \dots + C_m x_m$$

DIMENSION LEMMA 2.3 Let $m \geq 2$ and let

$$A = \bigcap_{i=1}^m f_i^{-1}(C_i)$$

$(f_1, \dots, f_m \in (K^n)', C_1, \dots, C_m \text{ absolutely convex in } K)$ be a coelementary set. If $\dim [f_1, \dots, f_m] < m$ then there is a $j \in \{1, \dots, m\}$ such that

$$A = \bigcap_{i \neq j} f_i^{-1}(C_i)$$

Proof. We may suppose (renumeration) that there exists a $k \in \{2, \dots, m\}$

and $\lambda_1, \dots, \lambda_k \in K \setminus \{0\}$ such that $\lambda_1 f_1 + \dots + \lambda_k f_k = 0$. The sets

$\lambda_1 C_1, \dots, \lambda_k C_k$ are linearly ordered by inclusion ; say that $\lambda_1 C_1$ is the largest among them. We claim that $f_1^{-1}(C_1) \supset \bigcap_{i=2}^k f_i^{-1}(C_i)$ (then the lemma follows). Indeed, let $x \in \bigcap_{i=2}^k f_i^{-1}(C_i)$. Then for each $i \in \{2, \dots, k\}$ we

have

$$-\lambda_1^{-1} \lambda_i f_i(x) \in \lambda_1^{-1} \lambda_i C_i \subset \lambda_1^{-1} \lambda_1 C_1 = C_1$$

so that

$$f_1(x) = -\lambda_1^{-1} \sum_{i=2}^k \lambda_i f_i(x) \in C_1 \quad \text{i.e. } x \in f_1^{-1}(C_1)$$

THEOREM 2.4 A subset of K^n is elementary if and only if it is coelementary.

Proof. Let $A = \bigcap_{i=1}^m f_i^{-1}(C_i)$ be a coelementary set. Thanks to Lemma 2.3 we may suppose that f_1, \dots, f_m are linearly independent elements of $(K^n)'$.

Extend f_1, \dots, f_m to a base f_1, \dots, f_n of $(K^n)'$ and set $C_i := K$ for

$i \in \{m+1, \dots, n\}$. Then $A = \bigcap_{i=1}^n f_i^{-1}(C_i)$. There exist $x_1, \dots, x_n \in K^n$ such that

$f_i(x_j) = \delta_{ij}$ ($i, j \in \{1, \dots, n\}$). It takes only simple verifications to prove

that $A = C_1 x_1 + \dots + C_n x_n$. If, conversely, A is an elementary set then,

by using Lemma 2.2, we obtain $A = C_1 x_1 + \dots + C_n x_n$ where x_1, \dots, x_n is a

base of K^n and C_1, \dots, C_n are absolutely convex in K . Define $f_1, \dots, f_n \in (K^n)'$

by the formulas $f_i(x_j) = \delta_{ij}$ ($i, j \in \{1, \dots, n\}$). One verifies directly

that $A = \bigcap_{i=1}^n f_i^{-1}(C_i)$, which finishes the proof.

COROLLARY 2.5

- (i) An elementary set is absolutely convex.
- (ii) Finite sums and finite intersections of elementary sets are elementary.
- (iii) The image and the preimage of an elementary set under a linear map are elementary.
- (iv) A onedimensional absolutely convex set is elementary. An absolutely convex set containing a hyperplane is elementary.
- (v) A nonzero elementary set is a finite direct sum of onedimensional absolutely convex sets.

Proof. For (i)-(iv), use Theorem 2.4 in combination with the definition of co(elementary) set. (v) is a consequence of Lemma 2.2.

Our next goal is to show (Theorem 2.9) that elementaricity of an absolutely convex set is in fact a property of its associated seminorm.

LEMMA 2.6 Let A, B be absolutely convex subsets of K^n . Then

$$p_A = p_B \iff A^i = B^i \iff A^e = B^e \iff A^i \subset B \subset A^e.$$

LEMMA 2.7 Let A, B $\subset K^n$ be absolutely convex, $A \cap B = (0)$. Then

$$(A \oplus B)^i = A^i \oplus B^i, (A \oplus B)^e = A^e \oplus B^e.$$

The (easy) proofs are left to the reader.

Remark. If in Lemma 2.7 we drop the condition $A \cap B = (0)$ we still

have $(A + B)^i = A^i + B^i$. However it is not true in general that

$$(A + B)^e = A^e + B^e \text{ (Example 5.4).}$$

LEMMA 2.8 Let A, B $\subset K^n$ be absolutely convex, $A^i \subset B \subset A^e$.

Then there exist $x_1, \dots, x_m \in B \setminus A^i$, $m \leq n$, such that $B = A^i + \text{co}(x_1, \dots, x_m)$.

Proof. Let x_1, \dots, x_m be a maximal p_A -orthogonal system in $B \setminus A^i$. Then

x_1, \dots, x_m are linearly independent so $m \leq n$. Clearly $A^i + \text{co}(x_1, \dots, x_m)$

$\subset B$. Conversely, let $x \in B \setminus A^i$. Then $\{x\}$ and $[x_1, \dots, x_m]$ are not

p_A -orthogonal so there exist $\lambda_1, \dots, \lambda_m \in K$ such that $p_A(x - \sum_{i=1}^m \lambda_i x_i) < 1$.
 As $1 = p_A(x) = p_A(\sum_{i=1}^m \lambda_i x_i) = \max_i |\lambda_i|$ and $x - \sum_{i=1}^m \lambda_i x_i \in A^i$ we find
 $x \in A^i + \text{co}(x_1, \dots, x_m)$.

THEOREM 2.9 Let $A, B \subset K^n$ be absolutely convex and $p_A = p_B$.

If A is elementary then so is B .

Proof. Let A be elementary, say, $A = C_1 x_1 \oplus \dots \oplus C_m x_m$

(where $x_1, \dots, x_m \in K^n$ and C_1, \dots, C_m are absolutely convex in K). From

Lemma 2.7 we infer that $A^i = C_1^i x_1 \oplus \dots \oplus C_m^i x_m$ is elementary.

As $A^i \subset B \subset A^e$ (Lemma 2.6) the set B is elementary by Lemma 2.8.

PROPOSITION 2.10 (Compare Example 5.4) If $A, B \subset K^n$ are elementary
then $(A + B)^e = A^e + B^e$.

Proof. Trivially, $A^e + B^e \subset (A + B)^e$. To prove the opposite inclusion it suffices to show that $A^e + B^e$ is edged. We may suppose $A \neq (0)$, $B \neq (0)$. Then (Lemma 2.7) A^e and B^e are (direct) sums of onedimensional edged sets. So $A^e + B^e$ is a finite sum of onedimensional edged sets. By the Dimension Lemma 2.2 it is a direct sum of edged sets, hence edged by Lemma 2.7.

LEMMA 2.11 If $A, B, C \subset K^n$ are absolutely convex, $C = A \oplus B$,
then $[A]$ is orthogonal to $[B]$ with respect to p_C .

Proof. Let $x \in [A]$, $y \in [B]$. It suffices to prove that $p_C(x+y) \geq p_C(x)$.

If $\lambda \in K$ and $x + y \in \lambda(A+B)$ then $x - \lambda a = \lambda b - y$ for some $a \in A$, $b \in B$.

Hence, $x - \lambda a \in [A] \cap [B] = (0)$ so that $x = \lambda a$ (and $y = \lambda b$). We therefore

have $p_C(x+y) = \inf \{ |\lambda| : x + y \in \lambda(A+B) \} \geq \inf \{ |\lambda| : x \in \lambda A \} \geq$

$\inf \{ |\lambda| : x \in \lambda(A+B) \} = p_C(x)$.

THEOREM 2.12 A nonzero absolutely convex set $A \subset K^n$ is elementary if
and only if $[A]$ has a p_A -orthogonal base.

Proof. Let A be elementary, say $A = C_1 x_1 \oplus \dots \oplus C_m x_m$ where $C_i x_i \neq (0)$ for each i . Clearly x_1, \dots, x_m span $[A]$ and, by Lemma 2.11, they are orthogonal for P_A . Conversely, suppose $[A]$ has a P_A -orthogonal base e_1, \dots, e_m . To show that A is elementary we may assume that A is edged (Theorem 2.9) so that $A = \{x : p_A(x) \leq 1\}$.

Set $C_i := \{\lambda \in K : |\lambda| p_A(e_i) \leq 1\}$ ($i \in \{1, \dots, m\}$). One verifies immediately that $A = C_1 e_1 + \dots + C_m e_m$.

COROLLARY 2.13 Let K be spherically complete.

- (i) Each absolutely convex subset of K^n is elementary.
- (ii) The sum of two edged sets in K^n is edged.

Proof. (i) Let $A \subset K^n$ be absolutely convex. By spherical completeness of K , $[A]$ has a p_A -orthogonal base [2]. Then use Theorem 2.12. To prove (ii) combine (i) and Proposition 2.10.

Remark. If K is not spherically complete not every absolutely convex subset of K^2 is elementary (Example 5.1)

§3 THE ULTRAMETRIC HELLY THEOREM

In this section we consider further consequences of the Dimension Lemma 2.2.

PROPOSITION 3.1 Let $Y = A_1 + \dots + A_m$ where A_1, \dots, A_m are absolutely convex subsets of K^n and where $m > n$. Then there is a set $V \subset \{1, \dots, m\}$, of n elements, such that $Y = \sum_{i \in V} A_i$.

Proof. It suffices to prove that Y is the sum of $m-1$ members of $\{A_1, \dots, A_m\}$. Suppose for each $j \in \{1, \dots, m\}$ we had

$$Y \neq \sum_{i \neq j} A_i$$

Then we could find x_1, \dots, x_m such that $x_j \in A_j \setminus \bigcup_{i \neq j} A_i$ for each $j \in \{1, \dots, m\}$. By the Dimension Lemma 2.2 we have

$$x_j \in \text{co}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m) \subset \bigcup_{i \neq j} A_i$$

for some j contradicting $x_j \in A_j \setminus \bigcup_{i \neq j} A_i$.

PROPOSITION 3.2 Let $Z = A_1 \cap \dots \cap A_m$ where A_1, \dots, A_m are absolutely convex subsets of K^n and $m > n$. Then there is a set $V \subset \{1, \dots, m\}$, of n elements, such that $Z = \bigcap_{i \in V} A_i$.

Proof. It suffices to derive a contradiction from the assumption

$$Z \neq \bigcap_{i \neq j} A_i \quad (j \in \{1, \dots, m\})$$

Choose $x_j \in (\bigcap_{i \neq j} A_i) \setminus A_j$ ($j \in \{1, \dots, m\}$). By the Dimension Lemma 2.2

$$x_j \in \text{co}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m)$$

for some j . If $i \neq j$ then $x_i \in \bigcap_{s \neq i} A_s \subset A_j$ so that

$$x_j \in \text{co}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m) \subset A_j$$

which is, indeed, a contradiction.

A subset C of K^n is convex if $a, b, c \in C$, $\lambda, \mu, \nu \in B(0, 1)$,

$\lambda + \mu + \nu = 1$ implies $\lambda a + \mu b + \nu c \in C$. The following proposition is well known and easy to prove.

PROPOSITION 3.3

- (i) Absolutely convex sets are convex. An additive coset of an absolutely convex set is convex.
- (ii) A convex set containing 0 is absolutely convex. A nonempty convex set is an additive coset of an absolutely convex set.

THEOREM 3.4 (ULTRAMETRIC CARATHÉODORY THEOREM) Let $X \subset K^n$. The smallest convex set containing X equals

$$\left\{ \sum_{i=1}^{n+1} \lambda_i x_i : x_1, \dots, x_{n+1} \in X, \lambda_1, \dots, \lambda_{n+1} \in B(0, 1), \sum_{i=1}^{n+1} \lambda_i = 1 \right\}$$

Proof. We may suppose that $X \neq \emptyset$ so let $x_0 \in X$. The convex hull of X equals $x_0 + \text{co}(X - x_0)$. Let $y \in x_0 + \text{co}(X - x_0)$. By Proposition 3.1 there exist $x_1, \dots, x_n \in X$ and $\lambda_1, \dots, \lambda_n \in B(0, 1)$ such that $y = x_0 + \lambda_1(x_1 - x_0) + \dots + \lambda_n(x_n - x_0)$. We see that $y = \mu_0 x_0 + \mu_1 x_1 + \dots + \mu_n x_n$ where $\mu_i \in B(0, 1)$ for each i and $\sum \mu_i = 1$.

The rest of the proof is straightforward.

THEOREM 3.5 (ULTRAMETRIC HELLY THEOREM)

Let A_1, \dots, A_m be convex sets in K^n and $m > n + 1$. Then there is a set $V \subset \{1, \dots, m\}$, of $n + 1$ elements, such that

$$\bigcap_{i=1}^m A_i = \bigcap_{i \in V} A_i$$

Proof. It suffices to prove that $\bigcap_{i=1}^m A_i$ is the intersection of $m - 1$ members of $\{A_1, \dots, A_m\}$. This is trivial if $A_1 \cap \dots \cap A_{m-1} = \emptyset$ so let $a \in A_1 \cap \dots \cap A_{m-1}$. Then $A_1 \cap \dots \cap A_{m-1} = a + \bigcap_{i=1}^{m-1} (A_i - a)$. By Proposition 3.2 ($m-1 > n$ and $A_i - a$ is absolutely convex) we have, say, $\bigcap_{i=1}^{m-1} (A_i - a) = \bigcap_{i=1}^{m-2} (A_i - a)$. Then $A_1 \cap \dots \cap A_{m-1} = a + \bigcap_{i=1}^{m-2} (A_i - a) = A_1 \cap \dots \cap A_{m-2}$.

Remark. A weaker form of Theorem 3.5 yields the 'classical' form of Helly's Theorem: let C be a finite collection of convex subsets of K^n and suppose that each $n + 1$ members of C have a nonempty intersection. Then $\bigcap C \neq \emptyset$.

§4 PSEUDOPOLAR SETS

DEFINITION 4.1 ([5]) An absolutely convex set $A \subset K^n$ is pseudopolar if for each $x \in K^n \setminus A$ there is an $f \in (K^n)'$ such that $f(x) \notin f(A)$.

Before going to study pseudopolarity let us first quote some results on polar sets. For $A \subset K^n$ we set ([3],[4])

$$A^{\circ} := \{f \in (K^n)' : |f(a)| \leq 1 \text{ for all } a \in A\}$$

$$A^{\circ\circ} := \{x \in K^n : |f(x)| \leq 1 \text{ for all } f \in A^{\circ}\}$$

A is a polar set if $A = A^{\circ\circ}$. The following proposition is easy to prove ([3]).

PROPOSITION 4.2^a Let $A \subset K^n$ be absolutely convex. Then $A \subset A^{\circ\circ}$, $A^{\circ\circ}$ is polar; it is the smallest polar set containing A. The following assertions are equivalent.

- (α) $A = A^{\circ\circ}$.
- (β) For each $x \in K^n \setminus A$ there is an $f \in (K^n)'$ with $|f(a)| \leq 1$ for each $a \in A$ and $|f(x)| > 1$.
- (γ) For each $x \in K^n \setminus A$ there is an $f \in (K^n)'$ with $f(x) \notin f(A)^e$.

In the same spirit we define for $A \subset K^n$

$$A^{\square} := \{f \in (K^n)' : |f(a)| < 1 \text{ for all } a \in A\}$$

$$A^{\square\square} := \{x \in K^n : |f(x)| < 1 \text{ for all } f \in A^{\square}\}$$

The following is easy to prove.

PROPOSITION 4.2^b. Let $A \subset K^n$ be absolutely convex. Then $A \subset A^{\square\square}$, $A^{\square\square}$ is pseudopolar; it is the smallest pseudopolar set containing A.

The following are equivalent.

- (α) $A = A^{\square\square}$.
- (β) For each $x \in K^n \setminus A$ there is an $f \in (K^n)'$ with $|f(a)| < 1$ for all $a \in A$ and $|f(x)| = 1$.
- (γ) A is pseudopolar.

PROPOSITION 4.3

- (i) A subset of K^n is polar if and only if it is edged.
- (ii) A polar set is pseudopolar.
- (iii) Intersections of pseudopolar sets are pseudopolar.
- (iv) Elementary sets are pseudopolar.

Proof. For (i) see [3], Theorem 4.7. The statements (ii) and (iii) follow from Propositions 4.2^{a,b}. For (iv), observe that an elementary set is coelementary (Theorem 2.4) and that a set of the form $f^{-1}(C)$ ($f \in (K^n)'$, $C \subset K$ absolutely convex) is pseudopolar.

COROLLARY 4.4. ([5]) If K is spherically complete each absolutely convex subset of K^n is pseudopolar.

Proof. Corollary 2.13 (i) and Proposition 4.3 (iv).

We shall derive several characterizations of pseudopolarity in Theorem 4.8. To this end we need the following three lemmas.

LEMMA 4.5. Let $A \subset K^n$ be an indecomposable absolutely convex set of dimension > 1 . Then $A^{\square\square} = A^e$.

Proof. Obviously $A^{\square\square} \subset A^{\circ\circ} = A^e$. Suppose we had an $x \in A^e \setminus A^{\square\square}$. Then there would exist an $f \in (K^n)'$ with $f(x) \notin f(A)$. We have the decomposition $K^n = Kx \oplus \text{Ker } f$. We claim that $A \subset (Kx \cap A) + (\text{Ker } f \cap A)$. (Then $A = (Kx \cap A) \oplus (\text{Ker } f \cap A)$ yielding a contradiction proving the lemma.) Let $a \in A$. According to the decomposition of K^n we have

$$a = \lambda x + v$$

for some $\lambda \in K$, $v \in \text{Ker } f$. If $|\lambda|$ were ≥ 1 then $f(x) = f(\lambda^{-1}a) \in f(A)$. Hence, $|\lambda| < 1$ and (as $x \in A^e$), $\lambda x \in A$ and also $v = a - \lambda x \in A$.

Remark. Not every indecomposable set is onedimensional (Example 5.1).

LEMMA 4.6. Let $A \subset K^n$ be absolutely convex, let $\alpha \in K$, $|\alpha| > 1$.

Then there exists an elementary set X for which $A \subset X \subset \alpha A$.

Proof. If the valuation is discrete we may take $X := A$ so assume that the valuation is dense. Choose $\beta \in K$, $|\alpha|^{-1} < |\beta| < 1$. Observe that $A^e \subset \beta\alpha A$. By an obvious extension of [2], Theorem 3.16 (i) the space $[A]$ has a $|\beta|$ -orthogonal base e_1, \dots, e_m with respect to P_A i.e. for

$\lambda_1, \dots, \lambda_m \in K$ we have

$$|\beta| \max_i p_A(\lambda_i e_i) \leq p_A(\sum_i \lambda_i e_i) \leq \max_i p_A(\lambda_i e_i)$$

For $i \in \{1, \dots, m\}$ we set $C_i := \{\lambda \in K: p_A(\lambda e_i) \leq |\beta|^{-1}\}$. Then each C_i is absolutely convex. We claim that

$$A \subset C_1 e_1 + \dots + C_m e_m \subset \alpha A$$

In fact, let $x \in A$, $x = \lambda_1 e_1 + \dots + \lambda_m e_m$. From

$$1 \geq p_A(x) \geq |\beta| \max_i p_A(\lambda_i e_i)$$

it follows that $\lambda_i \in C_i$ for each i so that $x \in C_1 e_1 + \dots + C_m e_m$.

Further, if $x = c_1 x_1 + \dots + c_m x_m$ where $c_i \in C_i$ for each i then

$$p_A(x) \leq \max_i p_A(c_i e_i) \leq |\beta|^{-1}. \text{ We see that } p_A(\beta x) \leq 1.$$

Consequently, $\beta x \in A^e \subset \beta \alpha A$. Hence, $x \in \alpha A$.

LEMMA 4.7. Let $A, B \subset K^n$ be absolutely convex, $A \cap B = (0)$.

Then $(A \oplus B)^{\square\square} = A^{\square\square} \oplus B^{\square\square}$.

Proof. We have $A^{\square\square} \subset A^e \subset [A]$, $B^{\square\square} \subset B^e$. Hence $A^{\square\square} \cap B^{\square\square} = (0)$.

Trivially, $A^{\square\square} + B^{\square\square} \subset (A+B)^{\square\square}$. Let $x \in (A \oplus B)^{\square\square}$. Then $x \in [A \oplus B] =$

$[A] \oplus [B]$ so that $x = s + t$ where $s \in [A]$, $t \in [B]$. We prove that

$s \in A^{\square\square}$, $t \in B^{\square\square}$. Let $f \in (K^n)'$. Then $f = f_1 + f_2$ where $f_1, f_2 \in (K^n)'$,

$f_1 = f$ on $[A]$, $f_1 = 0$ on $[B]$, $f_2 = f$ on $[B]$, $f_2 = 0$ on $[A]$. We have

$f(s) = f_1(s) = f_1(s+t) \in f_1(A+B) = f_1(A) = f(A)$. Similarly, $f(t) \in f(B)$.

Remark. The formula $(A+B)^{\square\square} = A^{\square\square} + B^{\square\square}$ is not true in general if

$A \cap B \neq (0)$ (Example 5.4).

THEOREM 4.8 Let $A \subset K^n$ be absolutely convex. The following are equivalent.

(α) A is pseudopolar.

(β) A is the direct sum of an elementary set and an edged set.

(γ) A is the intersection of an elementary set and an edged set.

(δ) There is an elementary set X such that $A = A^e \cap X$.

(ε) A is the intersection of a collection of elementary sets.

(ζ) There exist countably many elementary sets $V_1 \supset V_2 \supset \dots$
such that $A = \bigcap_{i=1}^{\infty} V_i$.

Proof. (α) ⇒ (β). We may suppose that A is not elementary. Take any splitting of A into a direct sum of indecomposables. After rearranging it has the form

$$A = S_1 \oplus \dots \oplus S_m \oplus T_1 \oplus \dots \oplus T_k$$

where $\dim S_i \leq 1$ for each i, $\dim T_j > 1$ for each j. From Lemma 4.7 one easily obtains that each summand is pseudopolar. Lemma 4.5 tells that $T_j = T_j^{\square\square} = T_j^e$ i.e. T_j is edged for each j. Then $S := S_1 \oplus \dots \oplus S_m$ is elementary, $T := T_1 \oplus \dots \oplus T_k$ is edged (Lemma 2.7) and $A = S \oplus T$.

(β) ⇒ (γ). Let $A = S \oplus T$ where S is elementary and T is edged. Then $S \oplus [T]$ is elementary, $[S] \oplus T$ is edged and $A = (S \oplus [T]) \cap ([S] \oplus T)$.

(γ) ⇒ (δ) Let $A = B \cap X$ where B is edged, X is elementary. Then

$$A = A^e \cap A = (A^e \cap B) \cap X = A^e \cap X.$$

(δ) ⇒ (ζ) To prove (ζ) we may assume that the valuation of K is dense.

Let $A = A^e \cap X$, where X is elementary. Let $\lambda_1, \lambda_2, \dots \in K$ be such that

$|\lambda_1| > |\lambda_2| > \dots, \lim_{i \rightarrow \infty} |\lambda_i| = 1$. By Lemma 4.6 there exists, for each i, an elementary set W_i such that

$$A \subset W_i \subset \lambda_i A$$

Then $A \subset \bigcap_i W_i \subset \bigcap_i \lambda_i A = A^e$. Now set $V_1 := W_1 \cap X, V_2 := W_1 \cap W_2 \cap X, \dots$

By Corollary 2.5(ii) the V_i are elementary. Clearly $V_1 \supset V_2 \supset \dots$ and

$$\bigcap_i V_i = A.$$

(ζ) ⇒ (ε) is trivial, (ε) ⇒ (α) follows from Proposition 4.3 (iv),

(iii).

Remark. Statement (c) of Theorem 4.8 may lead to the question as to which absolutely convex sets are the sum of countably many elementary sets. It is not hard to prove that for each absolutely convex set $A \subset K^n$ there exist $x_1, x_2, \dots \in A$ such that $A = \text{co } (x_1, x_2, \dots)$.

§5 EXAMPLES

In this section K is not spherically complete.

Let us call a norm $||| \cdot |||$ on a K -vector space E a special norm if

- (i) $x \in E, y \in E, x \perp y$ (with respect to $||| \cdot |||$) implies $x = 0$ or $y = 0$
- (ii) $\{|||x||| : x \in E\} = \{|\lambda| : \lambda \in K\}$.

For every $n \in \mathbb{N}$ there exist special norms on K^n (let $\overset{V}{K}$ be the spherical completion of K as a Banach space over K , take an n -dimensional subspace of $\overset{V}{K}$, see [2] for further details).

EXAMPLE 5.1 Let $n \geq 2$, let $||| \cdot |||$ be a special norm on K^n , let
 $A := \{x \in K^n : |||x||| < 1\}$. Then

- (i) A is indecomposable, $\dim A = n$,
- (ii) A is not elementary,
- (iii) A is not pseudopolar.

Proof (i). By Lemma 2.11 and the speciality of the norm A is indecomposable.

(iii). By Lemma 4.5 $A^{\square\square} = A^e = \{x \in K^n : |||x||| \leq 1\} \neq A$ (the latter inequality from property (ii) of a special norm). We see that A is not pseudopolar. (ii) follows from (i).

LEMMA 5.2. Let $(K^n, ||| \cdot |||)$ be as in Example 5.1, let D be a linear subspace of K^n , $D \neq (0)$, $D \neq K^n$. Let K^n/D be equipped with the natural quotient norm (again denoted $||| \cdot |||$), let $\pi : K^n \rightarrow K^n/D$ be the quotient map. Then

$$\pi(\{x \in K^n : ||x|| \leq 1\}) = \{y \in K^n/D : ||y|| < 1\}$$

Proof. For each $y \in K^n/D$ with $||y|| < 1$ there exists (by the properties of the quotient norm) an $x \in K^n$ with $\pi(x) = y$ and $||x|| < 1$. Conversely, let $x \in K^n$, $||x|| \leq 1$. If $||x|| < 1$ then $||\pi(x)|| \leq ||x|| < 1$, so assume $||x|| = 1$. Then $||\pi(x)|| \leq 1$. Suppose $||\pi(x)|| = 1$. Then $||\pi(x)|| = \inf \{||x-d|| : d \in D\} = ||x||$. We see that $||x-d|| \geq ||x||$ for all $d \in D$ i.e., $x \perp D$, a contradiction. It follows that $||\pi(x)|| < 1$.

LEMMA 5.3. Let K be separable. There exists a special norm $|| \cdot ||$ on K^3 and a vector $a \in K^3$, $a \neq 0$ such that the quotient norm $|| \cdot ||$ on K^3/Ka is again special.

Proof. We use [2], Example 5.E, p. 196. Let $z \in K^3/K$, $||z|| = 1$; choose $u \in K^3 \setminus Kz$. Then $E := p^{-1}([u, z])$ is a threedimensional space with a special norm and the norm on E/Ka (where $a := 1$) is also special.

EXAMPLE 5.4 Let $|| \cdot ||$ and a be as in Lemma 5.3, let $\pi : K^3 \rightarrow K^3/Ka$ be the quotient map. Set

$$A := \{x \in K^3 : ||x|| \leq 1\}$$

$$B := Ka$$

Then A, B are edged (hence pseudopolar).

(i) $\pi(A)$ is not pseudopolar (hence not edged).

(ii) $A + B$ is not pseudopolar (hence $(A+B)^{\square\square} \neq A^{\square\square} + B^{\square\square}$, $(A+B)^e \neq A^e + B^e$).

Proof.

(i) Lemma 5.2 and Example 5.1.

(ii) Let $b \in K^3$ such that $||\pi(b)|| = 1$. We prove that $b \notin A + B$, $b \in (A+B)^{\square\square}$. If b were in $A + B$ then $||\pi(b)|| < 1$ by Lemma 5.2. Hence $b \notin A + B$. Now let $f \in (K^3)'$; we prove that $f(b) \in f(A+B)$. This formule is trivial if $f(a)$ happens to be $\neq 0$ (since then $f(A+B) \supset f(Ka) = K$) so assume $f(a) = 0$ i.e. $f = g \circ \pi$ where $g \in (K^3/Ka)'$. By Lemma 5.2 we have

$\pi(A) = \{z \in K^3/Ka : ||z|| < 1\}$. As the quotient norm is special we have that $\pi(A)$ is indecomposable (Example 5.1) so that, by Lemma 4.5, $\pi(A)^{\square\square} = \pi(A)^e = \{z \in K^3/Ka : ||z|| \leq 1\}$. We see that $\pi(b) \in \pi(A)^{\square\square}$. It follows that $f(b) = g(\pi(b)) \in g(\pi(A)) = g(\pi(A+B)) = f(A+B)$.

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